Heaviside step function

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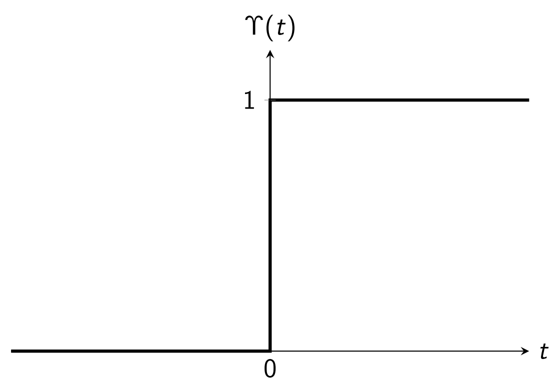
Version 1.0

Our first example of analog signal frequently appears in the study of electrical circuits, where at a moment $t = 0$, the circuit switches from an ”off” state associated with value 0 to an ”on” state associated with value 1. This signal is the Heaviside step function [1].

Definition 0.1 (Heaviside step function)
The **Heaviside step function** is the signal $\Upsilon \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined by:

$$\forall t \in \mathbb{R} \qquad \Upsilon(t) = \begin{cases} 0 & \text{if } t \in]-\infty, 0[\\ 1 & \text{if } t \in [0, +\infty[\end{cases}$$

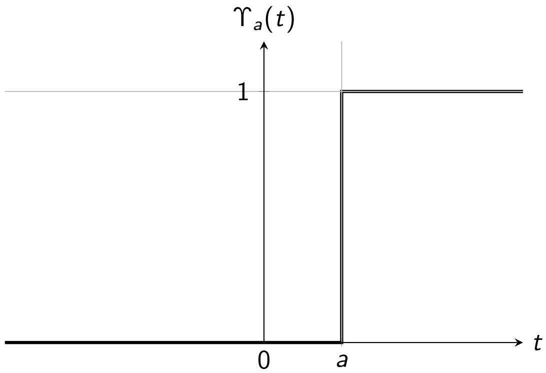
Using the notation for characteristic functions, we can write $\Upsilon = \chi_{[0,+\infty[}$ as well.



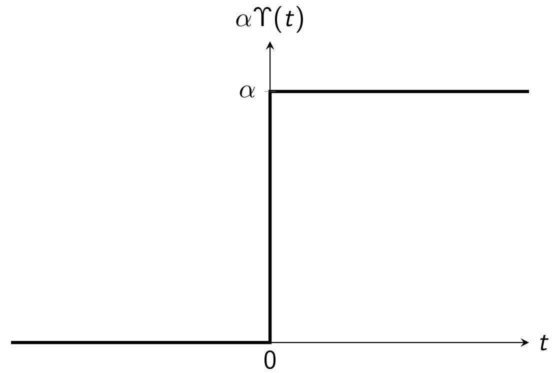
Remarks:

- An important property of Υ is the discontinuity in 0, which will cause problems when we study its differentiability in the next lecture.
- The Heaviside step function can be generalized by defining, for any $a \in \mathbb{R}$, the step centered in a: $\Upsilon_a : t \mapsto \Upsilon(t - a)$. By linearity, we can also define for any $\alpha \in \mathbb{R}$ the weighted step $\alpha\Upsilon : t \mapsto \alpha\Upsilon(t)$ which takes value α over $[0, +\infty[$.

[1] Oliver Heaviside (1850-1925), British physicist



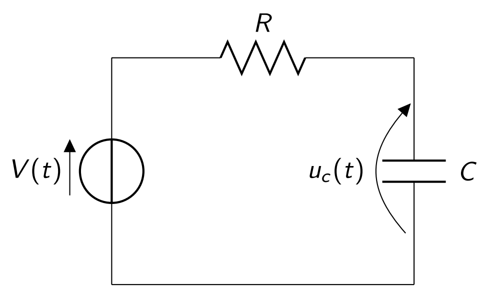
Shifted step function



Weighted step function

Definition 0.2 (Step response)
The **step response** of a system is the output corresponding to the Heaviside step function as input.

Example 0.1
We consider the following RC circuit system:



We denote R the resistance and C the capacity. The considered input is the voltage $V(t)$ of the source, and the output is the voltage $u_c(t)$ of the capacitor. This electrical system is governed by the following differential equation:

$$RCu_c'(t) + u_c(t) = V(t)$$

We add the physical constraint of voltage $u_c(t)$ being continuous over time. To determine the step response of this system, we have to solve this differential equation with $V = \Upsilon$. On one hand, the corresponding homogeneous differential equation

$$RCu_c'(t) + u_c(t) = 0 \qquad \Longleftrightarrow \qquad u_c'(t) + \frac{1}{RC}u_c(t) = 0$$

admits solutions of the form $u_c(t) = K \exp\left(-\frac{t}{RC}\right)$, with $K \in \mathbb{R}$. Since Υ is only differentiable over $\mathbb{R}^*$, we first look for a particular solution over $\mathbb{R}^*$, that we can then extend. It is clear that the derivative of restriction $\Upsilon_{|\mathbb{R}^*}$ is the zero function $0_{\mathbb{R}^*}$ over $\mathbb{R}^*$, making $\Upsilon_{|\mathbb{R}^*}$ a particular solution of this equation. Because of the discontinuity in 0, we start with two separate solutions over $]-\infty, 0[$ and $]0, +\infty[$:

$$u_c(t) = \begin{cases} K_1 \exp\left(-\frac{t}{RC}\right) & \text{if } t < 0 \\ 1 + K_2 \exp\left(-\frac{t}{RC}\right) & \text{if } t > 0 \end{cases}$$

Now we determine constants K_1 and K_2. The electrical circuit is off for $t \in]-\infty, 0[$ and we can assume that the capacitor is initially uncharged, implying $u_c(t) = 0$ for $t < 0$, thus $K_1 = 0$. Using the continuity of $u_c(t)$ in $t = 0$,

$$\lim_{t \to 0^-} u_c(t) = 0 = \lim_{t \to 0^+} u_c(t) = 1 + K_2$$

yields $K_2 = -1$. We conclude that the step response of this RC circuit is:

$$u_c(t) = \left(1 - \exp\left(-\frac{t}{RC}\right)\right) \Upsilon(t)$$